Constant mean curvature surfaces with boundary

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Lecture 1: Introduction and motivation

Question: What are the soap bubbles spanning a **circle**?

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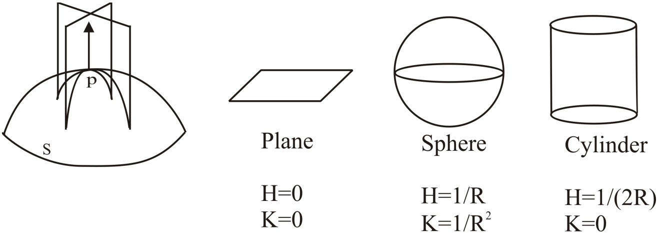
1. Introduction and motivation.
2. The tangency principle.
3. CMC compact surfaces with boundary.
4. The Dirichlet problem.

Introduction and motivation

- Classical Differential Geometry $\rightarrow$ submanifolds theory in three-dimensional manifolds.
- Variational problems.
- Problems with a physical origin.
- Differential geometry $\leftrightarrows$ PDEs

What is the curvature of a surface?

$$\lambda_1(p) = \max\{\text{normal curvatures at } p\}$$
$$\lambda_2(p) = \min\{\text{normal curvatures at } p\}.$$



- Gauss curvature: $K(p) = \lambda_1(p)\lambda_2(p)$.
- Mean curvature:

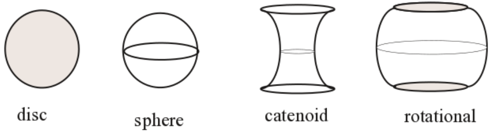
$$H(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2}.$$

Constant Mean Curvature surfaces $\equiv$ CMC surfaces

CMC surfaces...

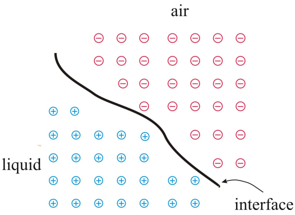
... minimize the area respect to some conditions:

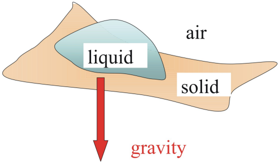
- a circular wire $\to$ planar disc (the boundary).
- blowing air obtaining soap bubbles $\to$ sphere (the volume).
- two coaxial circular wires $\to$ catenoid (the boundary).
- two coaxial circular wires and blowing air $\to$ rotational surfaces (boundary $+$ volume).

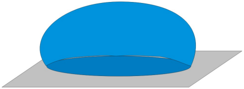


CMC surfaces are models of interfaces

An interface is the boundary of two homogenous systems of different physical/chemical properties.







liquid drop
resting on a plane

The energy E of this physical system consists of

i an energy from the surface tension: E_S.

ii an adhesion energy: E_A.

iii a gravitational energy: E_G

$$E = E_S + E_A + E_G$$

Laplace-Young equation:

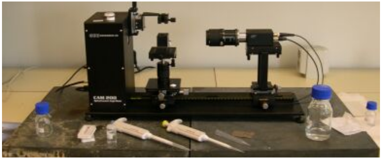
$$\boxed{(P_L - P_A) + k\ g = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\gamma = \underline{2H}\ \gamma.}$$

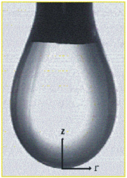
γ: surface tension coefficient.
R_1 and R_2 the principal curvatures of the interface S_{LA}.
$H = \frac{1}{2}\ \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$.

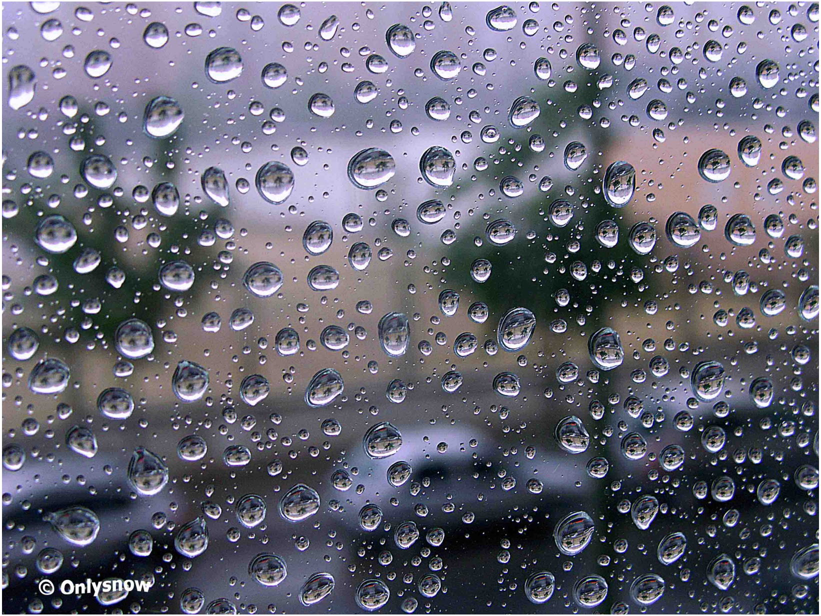
Under ideal conditions (no-gravity, constant pressures,...)

> *In equilibrium, the <u>interface</u> is a surface with <u>constant mean curvature</u> (CMC surface).*

- If $P_L - P_A = 0 \Rightarrow H = 0 \rightsquigarrow$ soap film.
- If $P_L - P_A = c \neq 0 \Rightarrow H = ct$: $\rightsquigarrow$ blowing air $\rightsquigarrow$ soap bubble.



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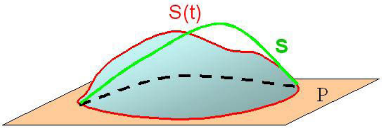
Solutions of a variational problem

Given a closed curve C...
Problem 1. Find the surface of least area spanning C.
Problem 2. Find the surface of least area spanning C enclosing a given volumen.



$$A'(0) = -2\int_M H\langle N, \xi\rangle, \quad \xi(p) = \frac{\partial X(p,t)}{\partial t}\Big|_{t=0}$$

$$V'(0) = -\int_M \langle N, \xi\rangle$$

Problem 1. $A'(0) = 0$ for any variation $\Leftrightarrow H = 0$.
Problem 2. $A'(0) + \lambda V'(0) = 0$ for any variation

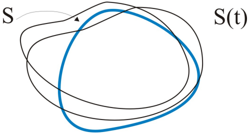
$$0 = A'(0) + \lambda V'(0) = -\int_M (2H + \lambda)\langle N, \xi \rangle = 0.$$

$\Leftrightarrow \exists \lambda$

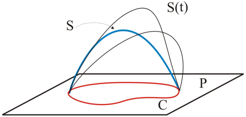
$$2H + \lambda = 0 \rightsquigarrow H = -\lambda/2 = c.$$

CMC surfaces are critical points of the area with respect to local deformations that <u>preserve the enclosed volume.</u>

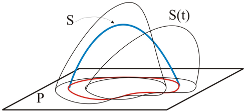
- Closed surfaces



- Preserving the boundary: $\partial S(t) = \partial S := C$.



- Keeping the boundary on a given support: $\partial S(t) \subset P$.



The contact angle is constant

Soap bubbles are round!

Soap bubbles minimize area enclosing a given volume: *among all closed surfaces enclosing a given volume, $\mathbb{S}^2$ is the only one with minimum area*.

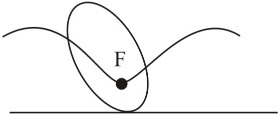
Problem: Given $C = \mathbb{S}^1$, find surfaces spanning C minimizing area for a given volume $\rightsquigarrow$ spherical caps

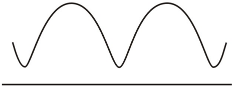
Examples of CMC surfaces

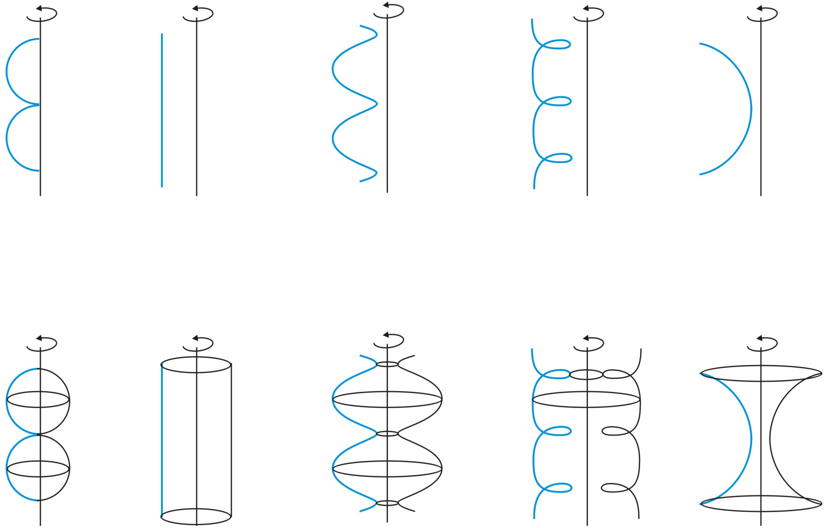
Rotational surfaces: $M = \{r(x)\cos\theta, r(x)\sin\theta, x) : x \in I, \theta \in \mathbb{R}\}$

$$1 + r'^2 - r'' = 2H(1 + r'^2)^{3/2} \Rightarrow \frac{d}{dx}\left(Hr^2 - \frac{r}{\sqrt{1 + r'^2}}\right) = 0.$$

$$Hr^2 - \frac{r}{\sqrt{1 + r'^2}} = c, \quad c \in \mathbb{R}.$$



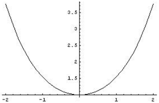


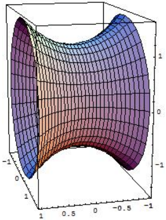


Catenoid and Riemann examples

Catenoid is the only rotational minimal surface:

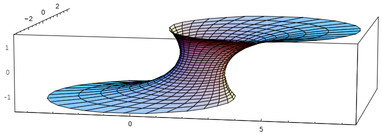
$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$





Riemann examples: minimal surfaces constructed by a uniparametric family of circles.

1. If the boundary are two concentric curves $\rightsquigarrow$ catenoid.
2. If the boundary are two non-coaxial circles in parallel planes $\rightsquigarrow$ Riemann example.



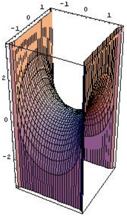
$H \neq 0 \rightsquigarrow$ rotational surface.

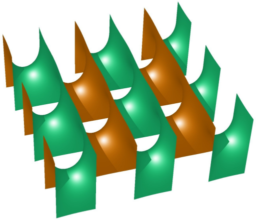
Translation surfaces: $z = f(x) + g(y)$

$$
\begin{aligned}
H = 0 \quad &\Rightarrow \quad (1 + g'^2) f'' + (1 + f'^2) g'' = 0 \\
&\Rightarrow \quad \frac{f''}{1 + f'^2} = -\frac{g''}{1 + g'^2} = c.
\end{aligned}
$$

$$
z = \log|\cosh(y)| - \log|\cosh(x)| = \log\left|\frac{\cosh(y)}{\cosh(x)}\right| \qquad (\textit{Scherk surface})
$$

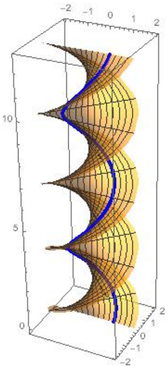
1. $H = 0$: plane and Scherk surface
2. $H = c \neq 0$: right circular cylinder





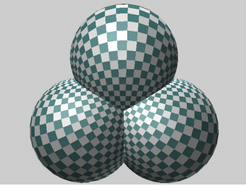
Ruled surfaces

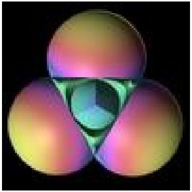
1. $H = 0$: plane and helicoid
2. $H = c \neq 0$: right circular cylinder



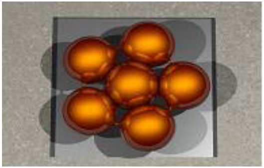
CMC closed surfaces

Wente (1984): 1-genus surface immersed in $\mathbb{R}^3$ with CMC





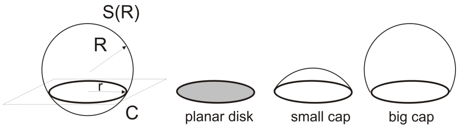
closed and not closed CMC surfaces (Kapouleas, Korevaar, Bobenko,...)



In the family of closed CMC surfaces, sphere is the only

1. of genus 0 (Hopf, ~ 50).
2. without self-intersections (Alexandrov, ~ 60).
3. stable (Barbosa-do Carmo, 1984).

Compact surfaces with boundary:
The simplest case $\partial S = \mathbb{S}^1$: discs and spherical caps.



Corollary

Planar disks and spherical caps are the only compact **rotational** *CMC surfaces spanning a circle.*

Planar discs and spherical caps are the only compact CMC surfaces with circular boundary that are

- Conjecture 1. topological discs
- Conjecture 2. embedded
- Conjecture 3. stable

Problems, questions, ...

Q1. Does the geometry of the boundary impose restrictions on the geometry of the surface?

Q2. Does the surface inherit the symmetry of its boundary?

Q3. (Plateau problem) Given a curve $C \subset \mathbb{R}^3$, $H \in \mathbb{R}$, does a surface S exist with $\partial S = C$ and (constant) mean curvature H?

Q4. (Dirichlet problem) Given a curve $C \subset \mathbb{R}^3$, $H \in \mathbb{R}$, does a graph S exist with $\partial S = C$ and (constant) mean curvature H?